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Free vibration analysis of a twisted beam using the dynamic stiffness method

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Abstract

An exact dynamics stiffness matrix is developed and subsequently used for free vibration analysis of a twisted beam whose flexural displacements are coupled in two planes. First the governing differential equations of motion of the twisted beam undergoing free natural vibration are derived using Hamilton's principle. Next the general solutions of these equations are obtained when the oscillatory motion of the beam is harmonic. This is followed by application of boundary conditions for displacements and forces, which essentially leads to the formation of the dynamics stiffness matrix of the twisted beam relating harmonically varying forces with harmonically varying displacements at its ends. The resulting dynamic stiffness matrix is used in connection with the Wittrick–Williams algorithm to compute natural frequencies and mode shapes of a twisted beam with cantilever end condition. These are compared with previously published results to confirm the accuracy of the method, and some conclusions are drawn.

Keywords: Free vibration; Twisted beam

1. Introduction

The free vibration analysis of twisted beams has aroused continuing research interest during the past half century. A small, but carefully selected sample of the relevant literature includes the work of Mandelson and Gendler (1951), Flax and Goland (1951), Rosard (1953), Troesch et al. (1954), Diprima and Handelman (1954), Zickel (1955), Carnegie (1959), Slyper (1962), Anliker and Troesch (1963), Dawson (1968), Carnegie and Thomas (1972), Lin (1977), Gupta and Rao (1978), Sisto and Chang (1984), Celep and Turham (1986), Rosen et al. (1987), Rosen (1991), Onipede et al. (1994), Liew and Lim (1994), Liew et al. (1994, 1995), Balhaddad and Onioede (1998) and Petrov and Geradin (1998). For background studies and historical development of the research, interested readers are referred to a survey paper by Rosen (1991), which gives an extensive bibliography on the subject. It is clear that prior to the development of the finite element method, a number of investigators (Troesch et al., 1954; Diprima and Handelman, 1954) attempted

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the problem of the free vibration analysis of twisted beams by relying on analytical solutions of the governing differential equations. These solutions yield natural frequencies and mode shapes of a twisted beam in the usual way by applying boundary conditions for displacements and forces. Despite the limitation that they focus only on a single structural element, these previous attempts, were and still are, useful contributions to the literature, and were no-doubt very significant at the time. The emergence of the finite element method has, of course, changed the attitude of many wishing to study this problem and as a result, has halted analytical developments on the subject. As a consequence, present day investigators are apparently unmotivated to seek analytical solutions of the type described above. For example, a twisted beam can be analysed approximately for its vibration characteristics using the finite element method by idealising it as a series of uniform untwisted beam elements. Each element is appropriately orientated to a set of global axis system. In this way, a twisted beam can be represented by a number of straight uniform beam elements and yet sufficiently accurate results can be obtained by increasing the number of elements. The simplicity as well as versatility of the finite element method has been extended by some investigators who have gone on to develop the element mass and stiffness matrices of twisted beams (Gupta and Rao, 1978; Sisto and Chang, 1984) and thus have enhanced the model accuracy significantly when compared with the simple idealization using straight elements. However, such finite element models will still not give exact results because the displacement function assumed for the twisted beam element is still inexact, with consequent errors in both stiffness and mass properties of the element.

Against this background it is now becoming progressively better known that there is a powerful alternative to the conventional finite element method, wherein the frequency dependent (exact) shape function, resulting from the solution of the governing differential equations, can be used (Williams and Wittrick, 1983; Williams, 1993; Banerjee, 1997). This is the method of the dynamic stiffness matrix, which has all the essential features of the finite element method and at the same time provides exact solutions to structural vibration problems. The method is undoubtedly superior to the traditional finite element method, particularly when higher natural frequencies and better accuracy of results are required. At present the range of applications of the dynamic stiffness method is somewhat limited to beams and a few restricted plate elements (Williams and Wittrick, 1983; Williams, 1993). Nevertheless this range is quite substantial. The method relies on only one single frequency dependent matrix called the dynamic stiffness matrix, which is obtained from the exact analytical solution of the governing differential equations of motion of the element undergoing free natural vibration. Once the initial assumptions on the displacement field have been made, the resulting differential equations are solved exactly and no further approximation is introduced. Thus the resulting element matrix features exactly the mass and stiffness properties of the element. The method of the dynamic stiffness matrix is well established and there are well-known software packages available (Anderson and Williams, 1987; Williams et al., 1991) based on the method. The element dynamic stiffness matrices in a structure can be assembled in a similar manner to that of the finite element method except that only one overall dynamic stiffness matrix is obtained (instead of separate mass and stiffness matrices) for the complete structure. It should be noted that when dealing with free vibration problems, the dynamic stiffness method leads to a transcendental eigenvalue problem generally solved using the Wittrick–Williams algorithm (Wittrick and Williams, 1971) whereas the finite element method usually leads to a linear eigenvalue problem. It is also significant that the accuracy of results using the finite element method depends on the number of elements used whereas the dynamic stiffness method has no such limitation because it accounts for an infinite number of natural frequencies of a vibrating structure and thus the results are independent of the number of elements used in the analysis. For instance, one single dynamic stiffness structural element can be used to determine any number of natural frequencies of the element to any desired accuracy. This is, of course, impossible in traditional finite element method.

Of particular interest in this work, is the development of an exact dynamic stiffness matrix for a twisted beam, and then to use it for the subsequent study of its free vibration characteristics. The dynamic stiffness formulation of a twisted beam is significantly more difficult than that of a Bernoulli–Euler beam because coupling between bending displacements in the two principal planes of bending occurs as a result of the twist. Starting from the fundamental assumptions of allowable displacements of the twisted beam, the kinetic and potential energy expressions are derived to formulate the Lagrangian. Hamilton's principle is then applied to derive the governing differential equations. As a by-product of the Hamiltonian formulation the natural boundary conditions lead to the expressions for the shear force and bending moment of the twisted beam. By assuming harmonic oscillation the governing differential equations are solved. Next the boundary conditions for bending displacement, bending rotation, shear force and bending moment are imposed in both planes and the arbitrary constants are eliminated from the general solution. This essentially recasts the ensuing equations in the form of a dynamic stiffness matrix of the twisted beam element, relating amplitudes of harmonically varying forces with amplitudes of harmonically varying displacements at its ends. Finally the resulting dynamic stiffness matrix is applied using the Wittrick–Williams algorithm (Wittrick and Williams, 1971) to obtain natural frequencies of a carefully chosen example. The results are compared with published results and some conclusions are drawn.

2. Theory

Fig. 1 shows the notation used for a twisted beam of length L in a right-handed Cartesian coordinate system. The global coordinate axes XYZ are shown at the left-hand end of the beam whereas the local coordinate axes xyz (in lower cases) which vary along the length, as a result of the twist, are shown on the right-hand side. The local y and global Y axes are coincident, both passing through the centroid, and are perpendicular to the beam cross-section, and therefore, represent the axis of twist of the beam. The rate of twist k is assumed to be constant along the length. Thus, if the twist is zero at the left-hand end, and φ (in radian) at the right-hand end, then $k = \varphi/L$. The two principal second moment of areas of the beam cross-section are taken to be I_{XX} and I_{ZZ}, respectively.

The derivation of the governing (partial) differential equations of motion of the twisted beam (see Fig. 1) undergoing free natural vibration is of some considerable complexity. This is achieved by applying Hamilton's principle (see Appendix A for details). The resulting differential equations are given by

$$EI_{ZZ}(u'''' + 2kw''' - 2k^2u'' - 2k^3w' + k^4u) + 2kEI_{XX}(w''' - 2ku'' - k^2w') + m\ddot{u} = 0 \tag{1}$$

$$EI_{XX}(w'''' - 2ku''' - 2k^2w'' + 2k^3u' + k^4w) - 2kEI_{ZZ}(u''' + 2kw'' - k^2u') + m\ddot{w} = 0 \tag{2}$$

where u and w are displacements in the x and z directions of a point lying on the centroidal axis and located at a distance y from the origin, m is the mass per unit length, E is the Young's modulus of the beam material so that EI_{XX} and EI_{ZZ} are the bending rigidities in the YZ and XZ planes, and a prime and an over dot represent differentiation with respect to distance y and time t, respectively.

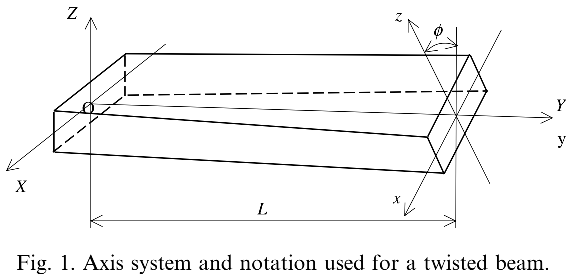

Fig. 1. Axis system and notation used for a twisted beam.

Eqs. (1) and (2) can also be written as

$$EI_{ZZ}u'''' - 2k^2(EI_{ZZ} + 2EI_{XX})u'' + k^4 EI_{ZZ}u + 2k(EI_{XX} + EI_{ZZ})w''' - 2k^3(EI_{XX} + EI_{ZZ})w' + m\ddot{u} = 0 \tag{3}$$

and

$$EI_{XX}w'''' - 2k^2(EI_{XX} + 2EI_{ZZ})w'' + k^4 EI_{XX}w - 2k(EI_{XX} + EI_{ZZ})u''' + 2k^3(EI_{XX} + EI_{ZZ})u' + m\ddot{w} = 0 \tag{4}$$

If harmonic variation of u and w with circular (angular) frequency ω is assumed then

$$\left.\begin{array}{l} u(y,t) = U(y)e^{i\omega t} \\ w(y,t) = W(y)e^{i\omega t} \end{array}\right\} \tag{5}$$

Substituting Eq. (5) into Eqs. (3) and (4) gives

$$EI_{ZZ}U'''' - 2k^2(EI_{ZZ} + 2EI_{XX})U'' + k^4 EI_{ZZ}U + 2k(EI_{XX} + EI_{ZZ})W''' - 2k^3(EI_{XX} + EI_{ZZ})W' - m\omega^2 U = 0 \tag{6}$$

and

$$EI_{XX}W'''' - 2k^2(EI_{XX} + 2EI_{ZZ})W'' + k^4 EI_{XX}W - 2k(EI_{XX} + EI_{ZZ})U''' + 2k^3(EI_{XX} + EI_{ZZ})U' - m\omega^2 W = 0 \tag{7}$$

Introducing the non-dimensional variable ξ (in place of y) where

$$\xi = yk = y\frac{\phi}{L} \tag{8}$$

Eqs. (6) and (7) can be combined into one differential equation by eliminating either U or W to obtain after simplification

$$D^8\Psi + 4D^6\Psi + (6 - a - b)D^4\Psi + \{4 + 6(a + b)\}D^2\Psi + (1 - a)(1 - b)\Psi = 0 \tag{9}$$

where

$$\Psi = U \ \text{ or } \ W \tag{10}$$

$$D = \frac{d}{d\xi} \tag{11}$$

and

$$a = \frac{m\omega^2}{EI_{XX}k^4} = \frac{m\omega^2 L^4}{EI_{XX}\phi^4} \tag{12}$$

$$b = \frac{m\omega^2}{EI_{ZZ}k^4} = \frac{m\omega^2 L^4}{EI_{ZZ}\phi^4} \tag{13}$$

The differential equation (9) is linear with constant coefficients so that the solution for Ψ (and hence for U and W) can be sought in the form

$$\Psi = e^{\lambda\xi} \tag{14}$$

Substituting Eq. (14) into Eq. (9) yields the auxiliary (or characteristic) equation

$$\lambda^8 + 4\lambda^6 + (6 - a - b)\lambda^4 + \{4 + 6(a + b)\}\lambda^2 + (1 - a)(1 - b) = 0 \tag{15}$$

The eighth order polynomial in λ above, can be reduced to a quartic as follows

$$\mu^4 + 4\mu^3 + (6 - a - b)\mu^2 + (4 + 6a + 6b)\mu + (1 - a)(1 - b) = 0 \tag{16}$$

where

$$\mu = \lambda^2 \quad \text{or} \quad \lambda = \pm\sqrt{\mu} \tag{17}$$

The roots of μ (and hence of λ) can now be obtained by using standard procedure (Press et al., 1986), for example, by factorizing the quartic of Eq. (16) into two quadratics. Note that the roots of μ (or λ) can be real or complex depending on the coefficients of the quartic.

Thus the solutions for U and W can be written as

$$U(\xi) = \sum_{j=1}^{8} A_j e^{\lambda_j \xi} = A_1 e^{\lambda_1 \xi} + A_2 e^{\lambda_2 \xi} + A_3 e^{\lambda_3 \xi} + A_4 e^{\lambda_4 \xi} + A_5 e^{\lambda_5 \xi} + A_6 e^{\lambda_6 \xi} + A_7 e^{\lambda_7 \xi} + A_8 e^{\lambda_8 \xi} \tag{18}$$

and

$$W(\xi) = \sum_{j=1}^{8} B_j e^{\lambda_j \xi} = B_1 e^{\lambda_1 \xi} + B_2 e^{\lambda_2 \xi} + B_3 e^{\lambda_3 \xi} + B_4 e^{\lambda_4 \xi} + B_5 e^{\lambda_5 \xi} + B_6 e^{\lambda_6 \xi} + B_7 e^{\lambda_7 \xi} + B_8 e^{\lambda_8 \xi} \tag{19}$$

where λ_j $(j = 1, 2, \ldots, 8)$ are the eight roots of the auxiliary equation (15) and A_j and B_j are two different sets of constants.

It can be shown by substituting Eqs. (18) and (19) into Eqs. (6) and (7) that the constants A_j and B_j are related as follows

$$B_j = \alpha_j A_j \tag{20a}$$

where

$$\alpha_j = \frac{\left\{\lambda_j^4 - 2\left(1 + 2\frac{b}{a}\right)\lambda_j^2 + (1 - b)\right\}}{2\left\{\left(1 + \frac{b}{a}\right)\lambda_j^3 - \left(1 + \frac{b}{a}\right)\lambda_j\right\}} \tag{20b}$$

With the help of Eqs. (18), (19), (20a) and (20b), the bending rotations θ_U and θ_W about the x and z axis are now respectively, given by (see Appendix A for the expressions of θ_U and θ_W)

$$\theta_U = \frac{dW}{dy} - \frac{\phi}{L}U(y) = \frac{\phi}{L}\frac{dW}{d\xi} - \frac{\phi}{L}U(\xi) = kW' - kU = k\sum_{j=1}^{8}\alpha_j A_j \lambda_j e^{\lambda_j \xi} - k\sum_{j=1}^{8} A_j e^{\lambda_j \xi} \tag{21}$$

$$\theta_W = -\frac{dU}{dy} - \frac{\phi}{L}W(y) = -\frac{\phi}{L}\frac{dU}{d\xi} - \frac{\phi}{L}W(\xi) = -kU' - kW = -k\sum_{j=1}^{8} A_j \lambda_j e^{\lambda_j \xi} - k\sum_{j=1}^{8}\alpha_j A_j e^{\lambda_j \xi} \tag{22}$$

The expressions for shear force and bending moment in the local axis system are given by (see Appendix A)

$$\begin{aligned} S_U &= EI_{ZZ}k^3[U''' + (2 + r)W'' - (1 + 2r)U' - rW] \\ &= EI_{ZZ}k^3\left[\sum_{j=1}^{8}\lambda_j^3 A_j e^{\lambda_j \xi} + (2 + r)\sum_{j=1}^{8}\alpha_j \lambda_j^2 A_j e^{\lambda_j \xi} - (1 + 2r)\sum_{j=1}^{8}\lambda_j A_j e^{\lambda_j \xi} - r\sum_{j=1}^{8}\alpha_j A_j e^{\lambda_j \xi}\right] \\ &= EI_{ZZ}k^3\sum_{j=1}^{8}\left\{\lambda_j^3 + (2 + r)\lambda_j^2\alpha_j - (1 + 2r)\lambda_j - r\alpha_j\right\}A_j e^{\lambda_j \xi} \end{aligned} \tag{23}$$

$$S_W = EI_{ZZ}k^3[rW''' - (2r+1)U'' - (2+r)W' + U]$$
$$= EI_{ZZ}k^3\left[r\sum_{j=1}^{8}\alpha_j\lambda_j^3A_j e^{\lambda_j\xi} - (1+2r)\sum_{j=1}^{8}\lambda_j^2A_j e^{\lambda_j\xi} - (2+r)\sum_{j=1}^{8}\alpha_j\lambda_jA_j e^{\lambda_j\xi} + \sum_{j=1}^{8}A_j e^{\lambda_j\xi}\right]$$
$$= EI_{ZZ}k^3\sum_{j=1}^{8}\left\{r\alpha_j\lambda_j^3 - (1+2r)\lambda_j^2 - (2+r)\alpha_j\lambda_j + 1\right\}A_j e^{\lambda_j\xi} \tag{24}$$

and

$$M_U = -EI_{XX}k^2(W'' - 2U' - W) = -EI_{XX}k^2\left[\sum_{j=1}^{8}\alpha_j\lambda_j^2A_j e^{\lambda_j\xi} - 2\sum_{j=8}^{8}\lambda_jA_j e^{\lambda_j\xi} - \sum_{j=1}^{8}\alpha_jA_j e^{\lambda_j\xi}\right]$$
$$= -EI_{XX}k^2\sum_{j=1}^{8}(\alpha_j\lambda_j^2 - 2\lambda_j - \alpha_j)A_j e^{\lambda_j\xi} \tag{25}$$

$$M_W = EI_{ZZ}k^2(U'' + 2W' - U) = EI_{ZZ}k^2\left[\sum_{j=1}^{8}\lambda_j^2A_j e^{\lambda_j\xi} + 2\sum_{j=1}^{8}\alpha_j\lambda_jA_j e^{\lambda_j\xi} - \sum_{j=1}^{8}A_j e^{\lambda_j\xi}\right]$$
$$= EI_{ZZ}k^2\sum_{j=1}^{8}(\lambda_j^2 + 2\alpha_j\lambda_j - 1)A_j e^{\lambda_j\xi} \tag{26}$$

where

$$r = \frac{EI_{XX}}{EI_{ZZ}} \tag{27}$$

The dynamic stiffness matrix of the twisted beam can now be obtained by applying the boundary condition for displacements and forces at its ends.

The boundary conditions for the bending displacement and bending rotation are: at the left-hand end

$$y = 0(\xi = 0): \quad U = U_1,\ \ W = W_1,\ \ \theta_U = \theta_{U_1},\ \ \theta_W = \theta_{W_1} \tag{28}$$

at the right-hand end

$$y = L(\xi = \phi = kL): \quad U = U_2,\ \ W = W_2,\ \ \theta_U = \theta_{U_2},\ \ \theta_W = \theta_{W_2} \tag{29}$$

The boundary conditions for the shear force and bending moment are: at the left-hand end

$$y = 0(\xi = 0): \quad S_U = S_{U_1},\ \ S_W = S_{W_1},\ \ M_U = M_{U_1},\ \ M_W = M_{W_1} \tag{30}$$

at the right-hand end

$$y = 0(\xi = \phi = kL): \quad S_U = -S_{U_2},\ \ S_W = -S_{W_2},\ \ M_U = -M_{U_2},\ \ M_W = -M_{W_2} \tag{31}$$

Substituting Eqs. (28) and (29) into Eqs. (18) and (19), Eqs. (21) and (22) respectively gives

$$U_1 = U(0) = \sum_{j=1}^{8}A_j \tag{32}$$

$$W_1 = W(0) = \sum_{j=1}^{8}\alpha_jA_j \tag{33}$$

$$\theta_{U_1} = \theta_U(0) = k\sum_{j=1}^{8}\alpha_j A_j \lambda_j - k\sum_{j=1}^{8} A_j \tag{34}$$

$$\theta_{W_1} = \theta_W(0) = -k\sum_{j=1}^{8} A_j \lambda_j - k\sum_{j=1}^{8} \alpha_j A_j \tag{35}$$

and

$$U_2 = U(\phi) = \sum_{j=1}^{8} A_j e^{\lambda_j \phi} \tag{36}$$

$$W_2 = W(\phi) = \sum_{j=1}^{8} \alpha_j A_j e^{\lambda_j \phi} \tag{37}$$

$$\theta_{U_2} = \theta_U(\phi) = k\sum_{j=1}^{8}\alpha_j \lambda_j A_j e^{\lambda_j \phi} - k\sum_{j=1}^{8} A_j e^{\lambda_j \phi} \tag{38}$$

$$\theta_{W_2} = \theta_W(\phi) = -k\sum_{j=1}^{8}\lambda_j A_j e^{\lambda_j \phi} - k\sum_{j=1}^{8} \alpha_j A_j e^{\lambda_j \phi} \tag{39}$$

Eqs. (32) and (39) can be written in matrix form as follows

$$\begin{bmatrix} U_1 \\ W_1 \\ \theta_{U_1} \\ \theta_{W_1} \\ U_2 \\ W_2 \\ \theta_{U_2} \\ \theta_{W_2} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} & R_{17} & R_{18} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} & R_{27} & R_{28} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} & R_{37} & R_{38} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} & R_{47} & R_{48} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} & R_{57} & R_{58} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} & R_{67} & R_{68} \\ R_{71} & R_{72} & R_{73} & R_{74} & R_{75} & R_{76} & R_{77} & R_{78} \\ R_{81} & R_{82} & R_{83} & R_{84} & R_{85} & R_{86} & R_{87} & R_{88} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{bmatrix} \tag{40}$$

or

$$\boldsymbol{\delta} = \boldsymbol{R}\boldsymbol{A} \tag{41}$$

where the elements of $\boldsymbol{R}$ (for$j = 1, 2, 3, \ldots, 8$) are given by

$$R_{1j} = 1 \tag{42}$$

$$R_{2j} = \alpha_j \tag{43}$$

$$R_{3j} = -k + k\alpha_j\lambda_j \tag{44}$$

$$R_{4j} = -k\alpha_j - k\lambda_j \tag{45}$$

$$R_{5j} = e^{\lambda_j \phi} \tag{46}$$

$$R_{6j} = \alpha_j e^{\lambda_j \phi} \tag{47}$$

$$R_{7j} = k\alpha_j\lambda_j e^{\lambda_j\phi} - k e^{\lambda_j\phi} \tag{48}$$

$$R_{8j} = -k\lambda_j e^{\lambda_j\phi} - k\alpha_j e^{\lambda_j\phi} \tag{49}$$

Substituting Eqs. (30) and (31) into Eqs. (23)–(26) gives

$$S_{U_1} = S_U(0) = EI_{ZZ}k^3 \sum_{j=1}^{8} \left\{ \lambda_j^3 + (2+r)\alpha_j\lambda_j^2 - (1+2r)\lambda_j - r\alpha_j \right\} A_j \tag{50}$$

$$S_{W_1} = S_W(0) = EI_{ZZ}k^3 \sum_{j=1}^{8} \left\{ r\alpha_j\lambda_j^3 - (1+2r)\lambda_j^2 - (2+r)\alpha_j\lambda_j + 1 \right\} A_j \tag{51}$$

$$M_{U_1} = M_U(0) = -EI_{XX}k^2 \sum_{j=1}^{8} (\alpha_j\lambda_j^2 - 2\lambda_j - \alpha_j) A_j \tag{52}$$

$$M_{W_1} = M_W(0) = EI_{ZZ}k^2 \sum_{j=1}^{8} (\lambda_j^2 + 2\alpha_j\lambda_j - 1) A_j \tag{53}$$

and

$$S_{U_2} = S_U(\phi) = -EI_{ZZ}k^3 \sum_{j=1}^{8} \left\{ \lambda_j^3 + (2+r)\alpha_j\lambda_j^2 - (1+2r)\lambda_j - r\alpha_j \right\} A_j e^{\lambda_j\phi} \tag{54}$$

$$S_{W_2} = S_W(\phi) = -EI_{ZZ}k^3 \sum_{j=1}^{8} \left\{ r\alpha_j\lambda_j^3 - (1+2r)\lambda_j^2 - (2+r)\alpha_j\lambda_j + 1 \right\} A_j e^{\lambda_j\phi} \tag{55}$$

$$M_{U_2} = M_U(\phi) = EI_{XX}k^2 \sum_{j=1}^{8} (\alpha_j\lambda_j^2 - 2\lambda_j - \alpha_j) A_j e^{\lambda_j\phi} \tag{56}$$

$$M_{W_2} = M_W(\phi) = -EI_{ZZ}k^2 \sum_{j=1}^{8} (\lambda_j^2 + 2\alpha_j\lambda_j - 1) A_j e^{\lambda_j\phi} \tag{57}$$

Eqs. (50)–(57) can be written in matrix form as follows

$$\begin{bmatrix} S_{U_1} \\ S_{W_1} \\ M_{U_1} \\ M_{W_1} \\ S_{U_2} \\ S_{W_2} \\ M_{U_2} \\ M_{W_2} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} & Q_{17} & Q_{18} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} & Q_{27} & Q_{28} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} & Q_{37} & Q_{38} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} & Q_{46} & Q_{47} & Q_{48} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} & Q_{56} & Q_{57} & Q_{58} \\ Q_{61} & Q_{62} & Q_{63} & Q_{64} & Q_{65} & Q_{66} & Q_{67} & Q_{68} \\ Q_{71} & Q_{72} & Q_{73} & Q_{74} & Q_{75} & Q_{76} & Q_{77} & Q_{78} \\ Q_{81} & Q_{82} & Q_{83} & Q_{84} & Q_{85} & Q_{86} & Q_{87} & Q_{88} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{bmatrix} \tag{58}$$

or

$$\boldsymbol{F} = \boldsymbol{QA} \tag{59}$$

where the elements of $\boldsymbol{Q}$ (for $j = 1, 2, \ldots, 8$) are given by

$$Q_{1j} = EI_{ZZ}\frac{\phi^3}{L^3}\left\{\lambda_j^3 + (2+r)\alpha_j\lambda_j^2 - (1+2r)\lambda_j - r\alpha_j\right\} \tag{60}$$

$$Q_{2j} = EI_{ZZ}\frac{\phi^3}{L^3}\left\{r\alpha_j\lambda_j^3 - (1+2r)\lambda_j^2 - (2+r)\alpha_j\lambda_j + 1\right\} \tag{61}$$

$$Q_{3j} = -EI_{XX}\frac{\phi^2}{L^2}(\alpha_j\lambda_j^2 - 2\lambda_j - \alpha_j) \tag{62}$$

$$Q_{4j} = EI_{ZZ}\frac{\phi^2}{L^2}(\lambda_j^2 + 2\alpha_j\lambda_j - 1) \tag{63}$$

$$Q_{5j} = -EI_{ZZ}\frac{\phi^3}{L^3}\left\{\lambda_j^3 + (2+r)\alpha_j\lambda_j^2 - (1+2r)\lambda_j - r\alpha_j\right\}e^{\lambda_j\phi} \tag{64}$$

$$Q_{6j} = -EI_{ZZ}\frac{\phi^3}{L^3}\left\{r\alpha_j\lambda_j^3 - (1+2r)\lambda_j^2 - (2+r)\alpha_j\lambda_j + 1\right\}e^{\lambda_j\phi} \tag{65}$$

$$Q_{7j} = EI_{XX}\frac{\phi^2}{L^2}(\alpha_j\lambda_j^2 - 2\lambda_j - \alpha_j)e^{\lambda_j\phi} \tag{66}$$

$$Q_{8j} = -EI_{ZZ}\frac{\phi^2}{L^2}(\lambda_j^2 + 2\alpha_j\lambda_j - 1)e^{\lambda_j\phi} \tag{67}$$

The constant vector $\boldsymbol{A}$ can now be eliminated from Eqs. (41) and (59) to give

$$\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{R}^{-1}\boldsymbol{\delta} = \boldsymbol{K}\boldsymbol{\delta} \tag{68}$$

where

$$\boldsymbol{K} = \boldsymbol{Q}\boldsymbol{R}^{-1} \tag{69}$$

is the required dynamic stiffness matrix.

When computing the dynamic stiffness matrix $\boldsymbol{K}$, it should be noted that the roots of μ and hence for λ, see Eqs. (15)–(17), can be complex and as a consequence, the elements of matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ can be complex. Therefore, the matrix inversion and multiplication steps of Eq. (69) must be carried out using complex arithmetic. The resulting dynamic stiffness matrix $\boldsymbol{K}$ will, of course, be symmetric and real, with imaginary parts of each element being zero.

Thus the force displacement relationship at the nodes of the harmonically vibrating twisted beam is given by

$$\begin{bmatrix} S_{U_1} \\ S_{W_1} \\ M_{U_1} \\ M_{W_1} \\ S_{U_2} \\ S_{W_2} \\ M_{U_2} \\ M_{W_2} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ & & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ & & & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ & & & & K_{55} & K_{56} & K_{57} & K_{58} \\ & S & Y & M & & K_{66} & K_{67} & K_{68} \\ & & & & & & K_{77} & K_{78} \\ & & & & & & & K_{88} \end{bmatrix} \begin{bmatrix} U_1 \\ W_1 \\ \theta_{U_1} \\ \theta_{W_1} \\ U_2 \\ W_2 \\ \theta_{U_2} \\ \theta_{W_2} \end{bmatrix} \tag{70}$$

It is now necessary to transform the above relationship to global coordinates using an appropriate transformation. Clearly the displacements and forces at the left-hand end of the twisted beam are already in

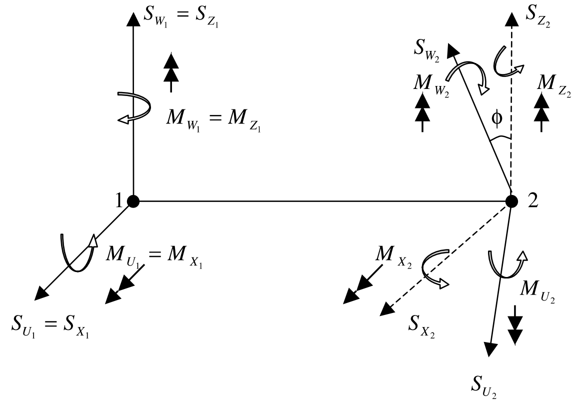

Fig. 2. Shear forces and bending moments at the ends of the twisted beam element shown in local and global coordinates.

global coordinates, whereas the corresponding displacements and forces at the right-hand end are in local coordinates (see Figs. 1 and 2).

Referring to Fig. 2, the shear forces and bending moments at the right-hand of the element can be resolved from global to local coordinates as follows:

$$S_{U_2} = S_{X_2}\cos\phi - S_{Z_2}\sin\phi \tag{71}$$

$$S_{W_2} = S_{X_2}\sin\phi + S_{Z_2}\cos\phi \tag{72}$$

$$M_{U_2} = M_{X_2}\cos\phi - M_{Z_2}\sin\phi \tag{73}$$

$$M_{W_2} = M_{X_2}\sin\phi + M_{Z_2}\cos\phi \tag{74}$$

Thus the relationships for the shear force and bending moment between the global and local coordinates at both ends of the beam element can be expressed as

$$\begin{bmatrix} S_{U_1} \\ S_{W_1} \\ M_{U_1} \\ M_{W_1} \\ S_{U_2} \\ S_{W_2} \\ M_{U_2} \\ M_{W_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s & 0 & 0 \\ 0 & 0 & 0 & 0 & s & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} S_{X_1} \\ S_{Z_1} \\ M_{X_1} \\ M_{Z_1} \\ S_{X_2} \\ S_{Z_2} \\ M_{X_2} \\ M_{Z_2} \end{bmatrix} \tag{75}$$

where

$$c = \cos\phi \tag{76}$$

and

$$s = \sin\phi \tag{77}$$

The displacements can be transformed from the global to local coordinates exactly in the same way as the forces by making use of the above transformation matrix T, given by

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s & 0 & 0 \\ 0 & 0 & 0 & 0 & s & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & 0 & 0 & s & c \end{bmatrix} \tag{78}$$

In this way the stiffness matrix of the twisted beam in global coordinates $\tilde{K}$ can now be formulated as

$$\tilde{K} = T'KT \tag{79}$$

where T' denotes the transpose of the transformation matrix T.

3. Application of the Wittrick–Williams algorithm

The dynamic stiffness matrix of Eq. (79) can now be used to compute the natural frequencies and mode shapes of twisted beams with various end conditions. A non-uniform twisted beam can also be analysed for its free vibration characteristics by idealizing it as an assemblage of many uniform twisted beams. An accurate and reliable method of calculating the natural frequencies and mode shapes of a structure using the dynamic stiffness method is to apply the well-known algorithm of Wittrick and Williams (1971) which has featured in numerous papers (Williams and Wittrick, 1983; Williams, 1993). Before applying the algorithm the dynamic stiffness matrices of all individual elements in a structure are to be assembled to form the overall dynamic stiffness matrix $\boldsymbol{K}_f$ of the final (complete) structure, which may, of course, consist of a single element. The algorithm monitors the Sturm sequence condition of $\boldsymbol{K}_f$ in such a way that there is no possibility of missing a frequency (or mode) of the structure. This is, of course, not possible in the conventional finite element method. The algorithm (unlike its proof) is very simple to use. However, the procedure is briefly summarized as follows.

Suppose that ω denotes the circular (or angular) frequency of a vibrating structure. Then according to the Wittrick–Williams algorithm (Wittrick and Williams, 1971), j, the number of natural frequencies passed, as ω is increased from zero to ω^*, is given by

$$j = j_0 + s\{\boldsymbol{K}_f\} \tag{80}$$

where $\boldsymbol{K}_f$, the overall dynamic stiffness matrix of the final structure whose elements all depend on ω, is evaluated at $\omega = \omega^*$; $s\{\boldsymbol{K}_f\}$ is the number of negative elements on the leading diagonal of $\boldsymbol{K}_f^{\Delta}$, $\boldsymbol{K}_f^{\Delta}$ is the upper triangular matrix obtained by applying the usual form of Gauss elimination to $\boldsymbol{K}_f$, and j_0 is the number of natural frequencies of the structure still lying between $\omega = 0$ and $\omega = \omega^*$ when the displacement components to which $\boldsymbol{K}_f$ corresponds are all zeros. (Note that the structure can still have natural frequencies when all its nodes are clamped, because exact member equations allow each individual member to displace between nodes with an infinite number of degrees of freedom, and hence infinite number of natural frequencies between nodes.) Thus

$$j_0 = \sum j_m \tag{81}$$

where j_m is the number of natural frequencies between $\omega = 0$ and $\omega = \omega^*$ for a component member with its ends fully clamped, while the summation extends over all members of the structure. For the element dynamic stiffness matrix developed in this paper, the clamped–clamped natural frequencies of an individual member are given by $\Delta = 0$, where Δ is the determinant of the matrix $\boldsymbol{Q}$ of Eq. (58) or Eq. (59). Thus, with the knowledge of Eqs. (80) and (81), it is possible to ascertain how many natural frequencies of a structure lie below an arbitrarily chosen trial frequency. This simple feature of the algorithm (coupled with the fact that successive trial frequencies can be chosen by the user to bracket a natural frequency) can be used to converge on any required natural frequency to any desired (or specified) accuracy.

4. Scope and limitations of the theory

The twisted beam considered in this paper is assumed to behave according to the Bernoulli–Euler theory in which the cross-sectional dimensions are assumed to be small compared to the length, and thus the effects of shear deformation and rotatory inertia are ignored. Also the beam has a constant rate of twist along its length and is assumed to exhibit coupling between bending displacements only. These displacements are considered to be uncoupled with torsional and/or extensional deformations. Also the cross-section of the beam is not allowed to warp. These assumptions are quite legitimate for many twisted beams with doubly symmetric cross-section, but they can be severe for many other (practical) twisted beams such as helicopter or turbine blades for which coupling between bending, torsional and extensional deformations and the rotational speed can have significant effects. The dynamic stiffness development of such complex twisted beams involves much more difficulty requiring additional insights. The theory presented in this paper is an essential prerequisite to study such problems, and is expected to pave the way for further research on dynamic stiffness developments of structural elements to a point where they can be applied directly to a wide range of problems in a design environment.

5. Results and discussion

The theory developed in this paper is applied to a cantilever blade taken from the literature (Rosen et al., 1987). For comparison of results, this particular reference was chosen because it uses a mathematical discretization approach based on principal as well as non-physical coordinates when investigating the free vibration characteristics of twisted beams. This approach gives more accurate results than the ones generally obtained from the conventional finite element method and thus provides an excellent basis for a direct comparison with the dynamic stiffness method. The angle of twist of the example blade is zero at the root and 40° at the root so that $\varphi = 2\pi/9$ radian. The structural and other properties used are: (i) $EI_{XX} = 2869.7$ Nm^2, (ii) $EI_{ZZ} = 57393.0$ Nm^2, (iii) $m = 34.47$ kg/m and (iv) $L = 3.048$ m. (Note that the mass per unit length m given on page 550 of the paper by Rosen et al. (1987) is 0.3447 kg/m, instead of 34.47 kg/m. This is surely a typographical error as evident from the corresponding value given in imperial unit in the paper.)

The first five natural frequencies of the blade obtained from the present theory are shown in Table 1 alongside the 20-segment results of Rosen et al. (1987) (see their Table 6 showing Mode A results). The

Table 1
Natural frequencies of a twisted blade with cantilever end condition

Frequency number	Natural frequency (rad/s)	
	Present theory	Rosen et al. (1987)
1	3.47173	3.47257
2	13.3465	13.2740
3	25.1707	25.2700
4	56.3716	56.3009
5	103.263	103.200

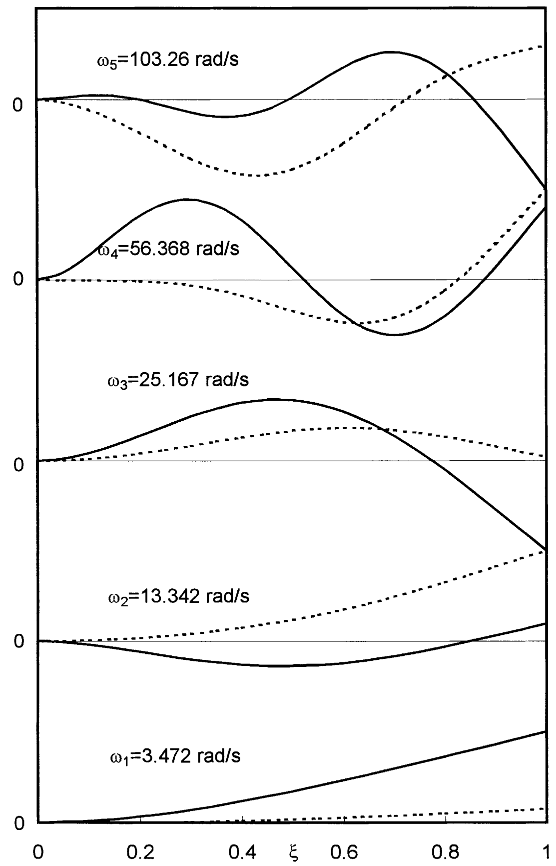

Fig. 3. The first five natural frequencies and mode shapes of a twisted blade. (—) Bending displacement in the *YZ* plane; (- - -) bending displacement in the *XY* plane.

agreement between the two sets of results is very good as can be seen. The corresponding mode shapes of the blade when using the present theory are shown in Fig. 3. These are also in good agreement with the mode shapes presented (partly in graphical and partly in tabular form) by Rosen et al. (1987). (Note that the opposite signs for the displacements in the *XY* plane are due to the differences in the notation and axis system chosen.) Clearly the mode shapes indicate very strong coupling between the displacements in two planes (i.e. the *XY* and *YZ* planes). The coupling is of course, induced by the pre-twist.

In order to achieve the same accuracy of results as given by the present theory, but by using the simple (untwisted) Bernoulli–Euler beam theory instead, it became very clear that a large number of uniform straight elements (with an appropriate orientation of each) are required. For instance, in order to obtain the six-figure accuracy in natural frequencies quoted in Table 1, it has been confirmed with the help of an established computer program called BUNVIS-RG (Anderson et al., 1986, 1987) that around 150 uniform straight elements are necessary to achieve the same results. Further investigation has shown that the approximate results using uniform straight elements converge almost parabolically with increasing number of elements. The results for the illustrative example using 10 and 20 elements and their parabolic limit are shown in Table 2. Clearly the parabolic limit is a very close approximation to the exact dynamic stiffness results shown in Table 1. This accords with an earlier investigation carried out by the author in the context of a rotating beam (Banerjee, 2000).

Table 2
Natural frequencies of the example twisted blade (Rosen et al., 1987) using straight untwisted beam elements

Frequency number	Natural frequency (rad/s)		
	10 elements	20 elements	Parabolic limit
1	3.47151	3.47177	3.47186
2	13.3822	13.3516	13.3414
3	25.0275	25.1318	25.1666
4	56.4517	56.3887	56.3677
5	103.098	103.227	103.270

Following the above analysis, a further study was undertaken to examine the effect of pre-twist on the natural frequencies and mode shapes of the blade. Thus, the original 40° pre-twist of the blade was altered by subsequently reducing it to 30°, 15°, and eventually to zero degree, without altering the rest of the data. The first five natural frequencies and mode shapes for these three cases are shown in Fig. 4. Note that the present theory does not allow the pre-twist to be set to exactly zero, but a small number, say, 10^{-6} can be safely used. This practically enables the present theory to converge back to Bernoulli–Euler theory giving

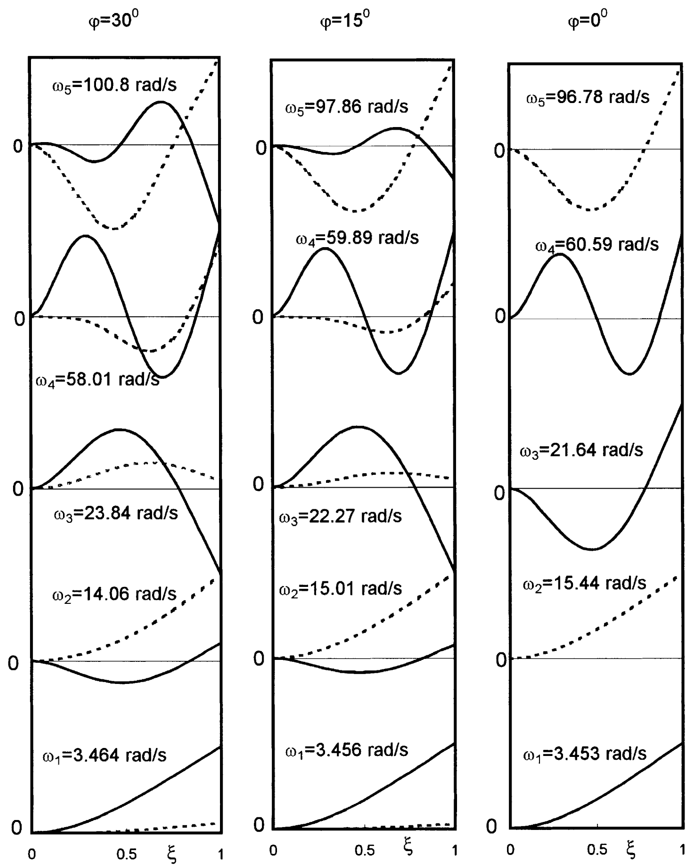

Fig. 4. The effect of pre-twist on the natural frequencies and mode shapes of a twisted beam. (—) Bending displacement in the *YZ* plane; (- - -) bending displacement in the *XY* plane.

uncoupled natural frequencies and mode shapes in the two planes. It is evident from Fig. 4 that the pretwist has a relatively minor effect on the natural frequencies, but has a much more pronounced effect on the mode shapes of the twisted blade.

6. Conclusions

Using Hamilton's principle the governing differential equations of motion of a twisted beam undergoing free natural vibration are derived and subsequently used to develop the dynamic stiffness matrix. The application of the dynamic stiffness matrix is demonstrated by numerical results that were obtained by using the Wittrick–Williams algorithm. The natural frequencies and mode shapes of an example (cantilever) blade with substantial twist have shown good agreement with published results. The effect of twist on natural frequencies and mode shapes has been further investigated. The results show that the twist has a more pronounced effect on mode shapes than on natural frequencies. It has been shown that when idealizing a twisted beam by using a number of untwisted beams, the parabolic limit gives an accurate estimate of exact results. The research presented in this paper can be used as an aid to validate the finite element and other approximate methods, and is expected to stimulate further research on the dynamic stiffness development of complex structural elements.

Appendix A

Derivation of the governing differential equations of motion of a twisted beam

The Hamiltonian mechanics is developed to derive the governing differential equations of motion of a twisted beam, having a uniform rate of twist and undergoing free natural vibration according to the Bernoulli–Euler beam theory. A set of allowable displacements and rotations is used as the starting point to form the system of direct and shearing strains. The expressions for strain energy and kinetic energy are then derived and subsequently used when applying Hamilton's principle.

In Fig. A1, $O(X,Y,Z)$ is an inertial frame, with OY along the line of centroids of the undeflected beam cross-sections. Let G be the centroid at $Y = y$, and Gx and Gz principal axes in bending of the cross-section. The two-dimensional axis system in the plane of the cross-section represented by $G(x,z)$ have a right-handed rotation ϕ about OY, so that the angle between Gx and OX (and also between Gz and OZ) is

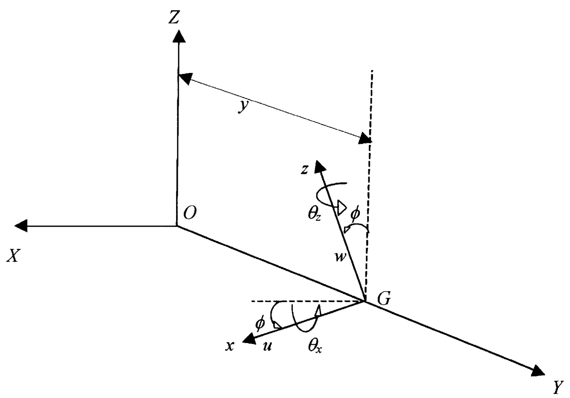

Fig. A1. Displacements and rotations of the centroid G at a distance y from the origin of the twisted beam shown in local coordinates.

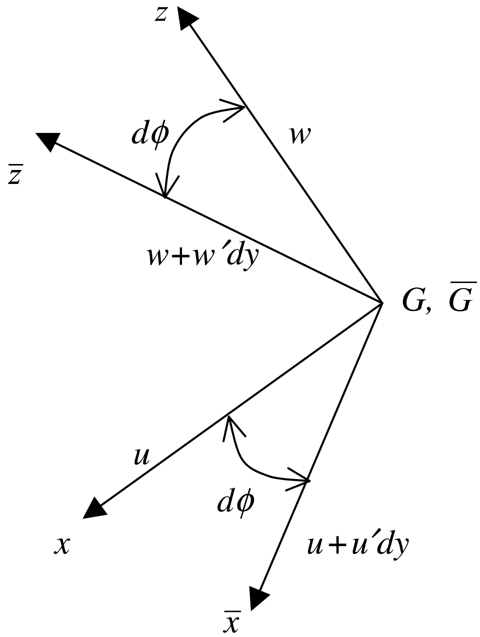

Fig. A2. Displacements of the centroids G and $\bar{G}$ at the left- and right-hand ends of an elemental length $\mathrm{d}y$ of the twisted beam shown in local coordinates.

ϕ as shown. This is the angle of twist at y so that the rate of twist k (which is assumed to be constant) is $\mathrm{d}\phi/\mathrm{d}y$.

Let the local displacements be u along Gx and w along Gz, and θ_x and θ_z be the rotations about the x- and z-axis respectively. Now consider an adjacent section at $Y = y + \mathrm{d}y$, and let $\bar{G}(\bar{x},\ \bar{y})$ be the corresponding axis system, see Fig. A2. Allowing for the relative rotation $\mathrm{d}\phi = k\,\mathrm{d}y$ of the element $\mathrm{d}y$, the relative displacements of $\bar{G}$ with respect to G, along Gx and Gz are respectively given by

$$\Delta x = (u + u'\,\mathrm{d}y)\cos \mathrm{d}\phi + (w + w'\,\mathrm{d}y)\sin \mathrm{d}\phi - u \tag{A.1}$$

and

$$\Delta z = (w + w'\,\mathrm{d}y)\cos \mathrm{d}\phi - (u + u'\,\mathrm{d}y)\sin \mathrm{d}\phi - w \tag{A.2}$$

To the first order, these are

$$\Delta x = (u' + kw)\,\mathrm{d}y \tag{A.3}$$

and,

$$\Delta z = (w' - ku)\,\mathrm{d}y \tag{A.4}$$

The rotations θ_x and θ_z of the cross-section about Gx and Gz are then respectively, given by

$$\theta_x = w' - ku \tag{A.5}$$

and

$$\theta_z = -u' - kw \tag{A.6}$$

A point P at (x, y, z) in the cross-section has displacement (u, v, w) along Gx, Gy, Gz where displacement in the Y-direction, V, is

$$V = -z\theta_x + x\theta_z \tag{A.7}$$

Substituting Eqs. (A.5) and (A.6) into Eq. (A.7) gives

$$V = -z(w' - ku) - x(u' + kw) \tag{A.8}$$

To determine the strain components, it is convenient to refer these local displacements to the inertial frame $O(X, Y, Z)$. It can be shown with the help of Fig. A1 that

$$x = X\cos\phi - Z\sin\phi \tag{A.9}$$

$$z = X\sin\phi + Z\cos\phi \tag{A.10}$$

and

$$U = u\cos\phi + w\sin\phi \tag{A.11}$$

$$W = -u\sin\phi + w\cos\phi \tag{A.12}$$

Substituting Eqs. (A.9) and (A.10) into Eq. (A.7) gives

$$V = -(X\sin\phi + Z\cos\phi)(w' - ku) - (X\cos\phi - Z\sin\phi)(u' + kw) \tag{A.13}$$

The shear strains γ_{XY} and γ_{YZ} are respectively, given by

$$\begin{aligned}\gamma_{XY} &= \frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y} \\ &= -\sin\phi(w' - ku) - \cos\phi(u' + kw) + u'\cos\phi - ku\sin\phi + w'\sin\phi + kw\cos\phi = 0\end{aligned} \tag{A.14}$$

and

$$\begin{aligned}\gamma_{YZ} &= \frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y} \\ &= -\cos\phi(w' - ku) + \sin\phi(u' + kw) - u'\sin\phi - ku\cos\phi + w'\cos\phi - kw\sin\phi = 0\end{aligned} \tag{A.15}$$

The shearing strains γ_{XY} and γ_{YZ} shown by Eqs. (A.14) and (A.15) are zeros as expected because of the assumption made in the Bernoulli–Euler beam theory.

The direct strain ε_y is obtained by the differentiating the expression for V in Eq. (A.13) with respect to y to give

$$\begin{aligned}\varepsilon_y &= \frac{\partial V}{\partial y} \\ &= -(X\cos\phi - Z\sin\phi)k(w' - ku) - (X\sin\phi + Z\cos\phi)(w'' - ku') + (X\sin\phi + Z\cos\phi)(u' + kw)k \\ &\quad - (X\cos\phi - Z\sin\phi)(u'' + kw')\end{aligned} \tag{A.16}$$

Eq. (A.16) with the help of Eqs. (A.9) and (A.10) becomes

$$\varepsilon_y = -x(u'' + 2kw' - k^2u) - z(w'' - 2ku' - k^2w) \tag{A.17}$$

Clearly

$$\varepsilon_X = \varepsilon_Z = \gamma_{XZ} = 0 \tag{A.18}$$

Thus the only non-zero strain is ε_y.

The strain energy $\mathscr{U}$ of the twisted beam in bending can now be expressed with the help of Eq. (A.17) as

$$\mathscr{U} = \frac{1}{2}\int_0^L \int_A \varepsilon_y^2 E\,\mathrm{d}A\,\mathrm{d}y = \frac{1}{2}EI_{XX}\int_0^L (w'' - 2ku' - k^2w)^2\,\mathrm{d}y + \frac{1}{2}EI_{zz}\int_0^L \left(u'' + 2kw' - k^2u\right)^2\mathrm{d}y \quad (\mathrm{A}.19)$$

where I_{XX} and I_{ZZ} are the principal second moment of areas of the beam cross-section about the X and Z axes and are respectively given by

$$I_{XX} = \int_A z^2\,\mathrm{d}A \quad (\mathrm{A}.20)$$

and

$$I_{ZZ} = \int_A x^2\,\mathrm{d}A \quad (\mathrm{A}.21)$$

The kinetic energy $\mathscr{T}$ of the twisted beam can be expressed as

$$\mathscr{T} = \frac{1}{2}\int_0^L m(\dot{u}^2 + \dot{w}^2)\,\mathrm{d}y \quad (\mathrm{A}.22)$$

Hamilton's principle states that $\delta\int_{t_1}^{t_2}(\mathscr{T} - \mathscr{U})\,\mathrm{d}t$ taken between arbitrary intervals of time t_1 and t_2 is stationary for a dynamic trajectory. Therefore

$$\delta\int_{t_1}^{t_2}(\mathscr{T} - \mathscr{U})\,\mathrm{d}t = 0 \quad (\mathrm{A}.23)$$

or for convenience

$$\delta\int_{t_1}^{t_2}(\mathscr{U} - \mathscr{T})\,\mathrm{d}t = 0 \quad (\mathrm{A}.24)$$

Substituting the expressions for $\mathscr{U}$ and $\mathscr{T}$ from Eqs. (A.19) and (A.22) into Eq. (A.24) and using the δ operator, one obtains

$$\int_{t_1}^{t_2}\int_0^L \{EI_{XX}\left(w'' - 2ku' - k^2w\right)\left(\delta w'' - 2k\,\delta u' - k^2\,\delta w\right) + EI_{ZZ}\left(u'' + 2kw' - k^2u\right)\left(\delta u'' + 2k\,\delta w' - k^2\,\delta u\right) - m\dot{u}\,\delta\dot{u} - m\dot{w}\,\delta\dot{w}\}\mathrm{d}y\,\mathrm{d}t = 0 \quad (\mathrm{A}.25)$$

Integrating by parts gives

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^L &\{EI_{XX}(w'''' - 2ku''' - k^2w'')\,\delta w + 2kEI_{XX}(w''' - 2ku'' - k^2w')\,\delta u - k^2EI_{XX}(w'' - 2ku' - k^2w)\,\delta w \\ &+ EI_{ZZ}(u'''' + 2kw''' - k^2u'')\,\delta u - 2kEI_{ZZ}(u''' + 2kw'' - k^2u')\,\delta w - k^2EI_{ZZ}(u'' + 2kw' - k^2u)\,\delta u \\ &+ m\ddot{u}\,\delta u + m\ddot{w}\,\delta w\}\,\mathrm{d}y\,\mathrm{d}t + \int_0^L[-m\dot{u}\,\delta u - m\dot{w}\,\delta w]_{t_1}^{t_2}\mathrm{d}y + \int_{t_1}^{t_2}\Big[EI_{XX}\left(w'' - 2ku' - k^2w\right)\delta w' \\ &- EI_{XX}\left(w''' - 2ku'' - k^2w'\right)\delta w + EI_{ZZ}\left(u'' + 2kw' - k^2u\right)\delta u' - EI_{ZZ}\left(u''' + 2kw'' - k^2u'\right)\delta u \\ &- 2kEI_{XX}\left(w'' - 2ku' - k^2w\right)\delta u + 2kEI_{ZZ}\left(u'' + 2kw' - k^2u\right)\delta w\Big]_0^L\mathrm{d}t = 0\end{aligned} \quad (\mathrm{A}.26)$$

Since δu and δw are completely arbitrary, the governing differential equations of motion in free vibration follow from the above equation as

$$EI_{ZZ}\left(u'''' + 2kw''' - 2k^2u'' - 2k^3w' + k^4u\right) + 2kEI_{XX}\left(w''' - 2ku'' - k^2w'\right) + m\ddot{u} = 0 \quad (\mathrm{A}.27)$$

and

$$EI_{XX}\left(w'''' - 2ku''' - 2k^2w'' + 2k^3u' + k^4w\right) - 2kEI_{ZZ}\left(u''' + 2kw'' - k^2u'\right) + m\ddot{w} = 0 \qquad (A.28)$$

From the natural boundary conditions, Eq. (A.26) gives the expressions for shear forces and bending moments as (Note that the signs have been reversed because the sign of the Lagrangian in Eq. (A.23) was reversed in Eq. (A.24))

$$S_x = EI_{ZZ}(u''' + 2kw'' - k^2u') + kEI_{XX}(w'' - 2ku' - k^2w) \qquad (A.29)$$

$$S_z = EI_{XX}(w''' - 2ku'' - k^2w') - kEI_{ZZ}(u'' + 2kw' - k^2u) \qquad (A.30)$$

$$M_x = -EI_{XX}(w'' - 2ku' - k^2w) \qquad (A.31)$$

and

$$M_z = EI_{ZZ}(u'' + 2kw' - k^2u) \qquad (A.32)$$

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